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Open quantum system of two coupled harmonic oscillators for application in deep inelastic heavy ion collisions†

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Abstract. On the basis of the theory of Lindblad for open quantum systems we derive master equations for a system consisting of two harmonic oscillators. The time dependence of expectation values, Wigner function and Weyl operator are obtained and discussed. The chosen system can be applied for the description of the charge and mass asymmetry degrees of freedom in deep inelastic collisions in nuclear physics.

1. Introduction

In recent years many experimental data have been measured in the field of deep inelastic heavy ion collisions in nuclear physics. A review has been given by Schröder and Huizenga (1984). The characteristic feature of these collisions is the binary character of the system, i.e. the final fragments have nearly the same masses as the initial nuclei. For the description of deep inelastic collisions one has to treat collective degrees of freedom explicitly. These are the relative motion of the nuclei, mass and charge exchange, the neck degree of freedom and surface vibrations of the fragments (Maruhn *et al* 1980). Another important feature of these reactions is the dissipation of energy and angular momentum out of the collective degrees of freedom into the intrinsic or single-particle degrees of freedom.

Deep inelastic collisions allow a description which is between two extreme theoretical approaches, namely in terms of transport theories and quantum mechanical collective theories. In the early theories the loss of kinetic energy is assumed as a direct process based on the exchange of nucleons between the nuclei, thus stressing independent-particle propagation and the stochastic random walk nature of the relaxation phenomenon (Weidenmüller 1980). In the later theories the loss of kinetic energy is seen as an indirect process: first collective modes, such as surface vibrations and giant resonances, are coherently excited and then damped due to the coupling of these modes to the remaining non-collective degrees of freedom (Dasso 1984). Therefore, the later theories assume that the relaxation phenomena are predominantly of coherent nature.

One method to introduce dissipation in a quantum mechanical description of deep inelastic collisions is to assume that the energy dissipation is similar to the loss of energy of a harmonic oscillator coupled with a large number of other harmonic

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oscillators. This mechanism can be simulated by a friction term of Kostin type in the Schrödinger equation (Kostin 1972). For charge and mass equilibration in deep inelastic collisions, such Schrödinger equations have been applied (Săndulescu *et al* 1981). But the Kostin-type Schrödinger equations have a non-linear character and are disadvantageous in the sense that they can only describe the dissipation from a single collective degree of freedom.

A more appropriate method of introducing dissipation into the quantum mechanical description of many coupled large scale collective modes is the axiomatic method of Lindblad (1976a, b). In this method, the simplest dynamics for the subsystem of the explicitly treated collective degrees of freedom is chosen, namely a semigroup of transformations which introduces a preferred direction in time and, therefore, can describe a genuinely irreversible process. It has been shown by Talkner (1986) that the assumption of a semigroup dynamics is only applicable in the limit of a weak coupling of the subsystem with its environment, e.g. for long relaxation times. However, in deep inelastic collisions the timescale of the subsystem is of the order of the relaxation time. Therefore, we have to consider the method of Lindblad in our applications as an axiomatic procedure for introducing dissipation processes in a simple but effective manner.

In a recent paper Săndulescu and Scutaru (1985) have applied the theory of Lindblad to the problem of the damping of a single collective coordinate in deep inelastic collisions. They have shown that various master equations for the damped quantum oscillator, used in the literature for the description of damped collective modes in deep inelastic collisions (Hofmann *et al* 1979, Hasse 1979, Spina and Weidenmüller 1984), are particular cases of the master equation derived by Lindblad.

In the present paper we extend the previous work of Gupta *et al* (1984) on the dynamics of the charge equilibration process in deep inelastic collisions and treat the damping of the proton and neutron asymmetry degrees of freedom with the method of Lindblad. The charge and mass distributions in di-nuclear systems can be described with continuous coordinates of the charge and neutron asymmetries defined by

$$\eta_Z = \frac{Z_1 - Z_2}{Z_1 + Z_2} \qquad \eta_N = \frac{N_1 - N_2}{N_1 + N_2}. \tag{1}$$

Here, $Z_1(N_1)$ and $Z_2(N_2)$ are the total charge numbers (neutron numbers) on the left-hand and right-hand side of a plane through the neck of the di-nuclear system (Maruhn *et al* 1980). Without damping, the charge and neutron asymmetry degrees of freedom are described by a wavefunction $\psi(\eta_Z, \eta_N, t)$, which is the solution of the following time-dependent Schrödinger equation:

$$H(\eta_Z, \eta_N)\psi(\eta_Z, \eta_N, t) = \mathrm{i}\hbar\, \partial\psi(\eta_Z, \eta_N, t)/\partial t \tag{2}$$

where the Hamiltonian of the model is given as ($|k_{ZN}| \leq (k_Z k_N)^{1/2}$):

$$H(\eta_Z, \eta_N) = -\frac{\hbar}{2B_{ZZ}} \frac{\partial^2}{\partial\eta_Z^2} - \frac{\hbar^2}{2B_{NN}} \frac{\partial^2}{\partial\eta_N^2} + \tfrac{1}{2}k_Z\eta_Z^2 + \tfrac{1}{2}k_N\eta_N^2 - k_{ZN}\eta_Z\eta_N. \tag{3}$$

The Hamiltonian has the simple structure of two coupled oscillators in the coordinates η_Z and η_N in order to keep the time development of the wavefunction analytically solvable. The string constants of the potential can be calculated with the liquid drop model for the sticking configuration of the nuclei, i.e. for the relative distance $R = R_1 + R_2$, where R_1 and R_2 are the radii of the two colliding nuclei (for more details see Gupta *et al* (1984)).

Hofmann *et al* (1979) have considered the coupling of the mass asymmetry coordinate to the charge asymmetry coordinate on the basis of a density operator formalism using a quantum master equation in a perturbative treatment. The coupling of the neutron and charge asymmetry coordinates has also been studied in the framework of a two-dimensional Fokker-Planck equation by Gross and Hartmann (1981), Schröder *et al* (1981), Birkelund *et al* (1982) and Merchant and Nörenberg (1982).

With the Hamiltonian (3) as an example, we study the damping of two coupled oscillators in the framework of the Lindblad theory. In order to have a formalism which is generally applicable, we give the following formulae in terms of the two general coordinates q_1 and q_2 instead of η_Z and η_N. In § 2 we present the equation of motion of the open quantum system of two oscillators in the Heisenberg picture. With this equation we derive the time dependence of the expectation values of the coordinates and momenta and their variances, as shown in § 3. The connection with the Wigner function and Weyl operator are discussed in § 4. Finally, in § 5, we demonstrate the time dependence of the various quantities for a simplified version of the model, where the decay constants can be calculated analytically.

2. The equation of motion in the Heisenberg picture

If $\tilde{\phi}_t$ is the dynamical semigroup describing the time evolution of the open quantum system in the Heisenberg picture, then the master equation is given for an operator A as follows (Lindblad 1976a, b):

$$\frac{\mathrm{d}\tilde{\phi}_t(A)}{\mathrm{d}t}=\tilde{L}(\tilde{\Phi}_t(A))=\frac{\mathrm{i}}{\hbar}[H,\tilde{\phi}_t(A)]+\frac{1}{2\hbar}\sum_j(V_j^+[\tilde{\Phi}_t(A),V_j]+[V_j^+,\tilde{\phi}_t(A)]V_j). \tag{4}$$

The operators H, V_j, V_j^+ $(j=1,2,3,4)$ are taken to be functions of the basic observables of the two quantum oscillators. The coordinates are q_1 and q_2, and the momenta p_1 and p_2 obey the usual commutation relations

$$[q_1,p_1]=\mathrm{i}\hbar I \qquad [q_2,p_2]=\mathrm{i}\hbar I$$

$$[q_1,q_2]=0 \qquad [p_1,p_2]=0 \qquad [q_1,p_2]=0 \qquad [q_2,p_1]=0.$$

In order to obtain an analytically solvable model, H is taken to be a polynomial of second degree in these basic observables and V_j, V_j^+ are taken to be polynomials of first degree. Then in the linear space spanned by q_1, q_2, p_1, p_2, there exist only four linearly independent operators $V_{j=1,2,3,4}$:

$$V_j=\sum_{\kappa=1}^{2}a_{j\kappa}p_\kappa+\sum_{\kappa=1}^{2}b_{j\kappa}q_\kappa \tag{5a}$$

where $a_{j\kappa}$, $b_{j\kappa}\in\mathbb{C}$ with $j=1,2,3,4$, and $\kappa=1,2$. Then it yields

$$V_j^+=\sum_{\kappa=1}^{2}a_{j\kappa}^*p_\kappa+\sum_{\kappa=1}^{2}b_{j\kappa}^*q_\kappa \tag{5b}$$

where $a_{j\kappa}^*$, $b_{j\kappa}^*$ are the complex conjugates of $a_{j\kappa}$, $b_{j\kappa}$.

The Hamiltonian H is chosen in the form of two coupled oscillators

$$H=\sum_{\kappa=1}^{2}\left(\frac{1}{2m_\kappa}p_\kappa^2+\frac{m_\kappa\omega_\kappa^2}{2}q_\kappa^2\right)+\kappa_{12}p_1p_2+\tfrac{1}{2}\sum_{\kappa_1,\kappa_2=1}^{2}\mu_{\kappa_1\kappa_2}(p_{\kappa_1}q_{\kappa_2}+q_{\kappa_2}p_{\kappa_1})+\nu_{12}q_1q_2. \tag{6}$$

Inserting the Hamiltonian H and the operators V_j and V_j^+ into (4) we obtain

$$\tilde{L}(A)=\tilde{L}_1(A)+\tilde{L}_2(A)+\tilde{L}_{12}(A) \tag{7}$$

where $\tilde{L}_1$, $\tilde{L}_2$ and $\tilde{L}_{12}$ are given as ($\kappa=1,2$)

$$\begin{aligned}\tilde{L}_\kappa(A)=&\frac{i}{\hbar}[H_{0\kappa},A]-\frac{1}{\hbar^2}D_{p_\kappa p_\kappa}[q_\kappa,[q_\kappa,A]]\\&-\frac{1}{\hbar^2}D_{q_\kappa q_\kappa}[p_\kappa,[p_\kappa,A]]+\frac{1}{\hbar^2}D_{p_\kappa q_\kappa}[q_\kappa,[p_\kappa,A]]\\&+\frac{1}{\hbar^2}D_{q_\kappa p_\kappa}[p_\kappa,[q_\kappa,A]]+\frac{i}{2\hbar}(\lambda_{\kappa\kappa}-\mu_{\kappa\kappa})([A,p_\kappa]q_\kappa+q_\kappa[A,p_\kappa])\\&-\frac{i}{2\hbar}(\lambda_{\kappa\kappa}+\mu_{\kappa\kappa})([A,q_\kappa]p_\kappa+p_\kappa[A,q_\kappa])\end{aligned} \tag{8}$$

$$\begin{aligned}\tilde{L}_{12}(A)=&-\hbar^{-2}D_{p_1p_2}([q_1,[q_2,A]]+[q_2,[q_1,A]])\\&-\hbar^{-2}D_{q_1q_2}([p_1,[p_2,A]]+[p_2,[p_1,A]])\\&+\hbar^{-2}D_{p_1q_2}([q_1,[p_2,A]]+[p_2,[q_1,A]])\\&+\hbar^{-2}D_{q_1p_2}([p_1,[q_2,A]]+[q_2,[p_1,A]])\\&+\tfrac{1}{2}i\hbar^{-1}(\alpha_{12}-\kappa_{12})([A,p_1]p_2+p_2[A,p_1])\\&-\tfrac{1}{2}i\hbar^{-1}(\alpha_{12}+\kappa_{12})([A,p_2]p_1+p_1[A,p_2])\\&+\tfrac{1}{2}i\hbar^{-1}(\beta_{12}-\nu_{12})([A,q_1]q_2+q_2[A,q_1])\\&-\tfrac{1}{2}i\hbar^{-1}(\beta_{12}+\nu_{12})([A,q_2]q_1+q_1[A,q_2])\\&+\tfrac{1}{2}i\hbar^{-1}(\lambda_{12}-\mu_{12})([A,p_1]q_2+q_2[A,p_1])\\&-\tfrac{1}{2}i\hbar^{-1}(\lambda_{12}+\mu_{12})([A,q_2]p_1+p_1[A,q_2])\\&+\tfrac{1}{2}i\hbar^{-1}(\lambda_{21}-\mu_{21})([A,p_2]q_1+q_1[A,p_2])\\&-\tfrac{1}{2}i\hbar^{-1}(\lambda_{21}+\mu_{21})([A,q_1]p_2+p_2[A,q_1]).\end{aligned} \tag{9}$$

Here, we used the following abbreviations ($\kappa=1,2$):

$$\begin{aligned}&H_{0\kappa}=\tfrac{1}{2}m_\kappa^{-1}p_\kappa^2+\tfrac{1}{2}m_\kappa\omega_\kappa^2q_\kappa^2\\&D_{q_\kappa q_\mu}=D_{q_\mu q_\kappa}=\tfrac{1}{2}\hbar\,\mathrm{Re}(\boldsymbol{a}_\kappa^*\boldsymbol{a}_\mu)\\&D_{p_\kappa p_\mu}=D_{p_\mu p_\kappa}=\tfrac{1}{2}\hbar\,\mathrm{Re}(\boldsymbol{b}_\kappa^*\boldsymbol{b}_\mu)\\&D_{q_\kappa p_\mu}=D_{p_\mu q_\kappa}=-\tfrac{1}{2}\hbar\,\mathrm{Re}(\boldsymbol{a}_\kappa^*\boldsymbol{b}_\mu)\\&\alpha_{12}=-\alpha_{21}=-\mathrm{Im}(\boldsymbol{a}_1^*\boldsymbol{a}_2)\\&\beta_{12}=-\beta_{21}=-\mathrm{Im}(\boldsymbol{b}_1^*\boldsymbol{b}_2)\\&\lambda_{\kappa\mu}=-\mathrm{Im}(\boldsymbol{a}_\kappa^*\boldsymbol{b}_\mu).\end{aligned} \tag{10}$$

The scalar products are formed with the vectors $\boldsymbol{a}_\kappa$, $\boldsymbol{b}_\kappa$ and their complex conjugates $\boldsymbol{a}_\kappa^*$, $\boldsymbol{b}_\kappa^*$. The vectors have the components

$$\boldsymbol{a}_\kappa=(a_{1\kappa},a_{2\kappa},a_{3\kappa},a_{4\kappa})$$

$$\boldsymbol{b}_\kappa=(b_{1\kappa},b_{2\kappa},b_{3\kappa},b_{4\kappa}).$$

Now, as a consequence of the definitions (10) of the phenomenological constants which appear in $\tilde{L}(A)$ and of the positivity of the matrix formed by the four vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{b}_1, \boldsymbol{b}_2$, it follows that the principal minors of this matrix are positive or zero. This matrix is given by

$$\frac{1}{2}\hbar\begin{pmatrix} \boldsymbol{a}_1^*\boldsymbol{a}_1 & \boldsymbol{a}_1^*\boldsymbol{a}_2 & \boldsymbol{a}_1^*\boldsymbol{b}_1 & \boldsymbol{a}_1^*\boldsymbol{b}_2 \\ \boldsymbol{a}_2^*\boldsymbol{a}_1 & \boldsymbol{a}_2^*\boldsymbol{a}_2 & \boldsymbol{a}_2^*\boldsymbol{b}_1 & \boldsymbol{a}_2^*\boldsymbol{b}_2 \\ \boldsymbol{b}_1^*\boldsymbol{a}_1 & \boldsymbol{b}_1^*\boldsymbol{a}_2 & \boldsymbol{b}_1^*\boldsymbol{b}_1 & \boldsymbol{b}_1^*\boldsymbol{b}_2 \\ \boldsymbol{b}_2^*\boldsymbol{a}_1 & \boldsymbol{b}_2^*\boldsymbol{a}_2 & \boldsymbol{b}_2^*\boldsymbol{b}_1 & \boldsymbol{b}_2^*\boldsymbol{b}_2 \end{pmatrix}$$

$$= \begin{pmatrix} D_{q_1q_1} & D_{q_1q_2}-(\mathrm{i}\hbar/2)\alpha_{12} & -D_{q_1p_1}-(\mathrm{i}\hbar/2)\lambda_{11} & -D_{q_1p_2}-(\mathrm{i}\hbar/2)\lambda_{12} \\ D_{q_2q_1}-(\mathrm{i}\hbar/2)\alpha_{21} & D_{q_2q_2} & -D_{q_2p_1}-(\mathrm{i}\hbar/2)\lambda_{21} & -D_{q_2p_2}-(\mathrm{i}\hbar/2)\lambda_{22} \\ -D_{p_1q_1}+(\mathrm{i}\hbar/2)\lambda_{11} & -D_{p_1q_2}+(\mathrm{i}\hbar/2)\lambda_{21} & D_{p_1p_1} & D_{p_1p_2}-(\mathrm{i}\hbar/2)\beta_{12} \\ -D_{p_2q_1}+(\mathrm{i}\hbar/2)\lambda_{12} & -D_{p_2q_2}+(\mathrm{i}\hbar/2)\lambda_{22} & D_{p_2p_1}-(\mathrm{i}\hbar/2)\beta_{21} & D_{p_2p_2} \end{pmatrix}. \tag{11}$$

For example, we derive the following condition from the positivity of (11):

$$D_{q_1q_1}D_{q_2q_2}-(D_{q_1q_2})^2 \geqslant \tfrac{1}{4}\hbar^2\alpha_{12}^2.$$

This inequality and the corresponding ones derived from (11) are constraints imposed on the phenomenological constants by the fact that $\tilde{\phi}_t$ is a dynamical semigroup (Lindblad 1976a, b).

3. The time dependence of expectation values

The time-dependent expectation values of self-adjoint operators A and B can be written with the density operator ρ, describing the initial state of the quantum system, as follows:

$$\begin{aligned} m_A(t) &= \mathrm{Tr}(\rho\tilde{\phi}_t(A)) \\ \sigma_{AB}(t) &= \tfrac{1}{2}\,\mathrm{Tr}(\rho\tilde{\phi}_t(AB+BA)). \end{aligned} \tag{12}$$

In the following we denote the vector with the four components $m_{q_1}(t)$, $m_{q_2}(t)$, $m_{p_1}(t)$ and $m_{p_2}(t)$ by $\boldsymbol{m}(t)$ and the following 4×4 matrix by $\hat{\sigma}(t)$:

$$\hat{\sigma}(t) = \begin{pmatrix} \sigma_{q_1q_1} & \sigma_{q_1q_2} & \sigma_{q_1p_1} & \sigma_{q_1p_2} \\ \sigma_{q_2q_1} & \sigma_{q_2q_2} & \sigma_{q_2p_1} & \sigma_{q_2p_2} \\ \sigma_{p_1q_1} & \sigma_{p_1q_2} & \sigma_{p_1p_1} & \sigma_{p_1p_2} \\ \sigma_{p_2q_1} & \sigma_{p_2q_2} & \sigma_{p_2p_1} & \sigma_{p_2p_2} \end{pmatrix}. \tag{13}$$

Then via direct calculation of $\tilde{L}(q_\kappa)$ and $\tilde{L}(p_\kappa)$ we obtain

$$\mathrm{d}\boldsymbol{m}/\mathrm{d}t = \hat{Y}\boldsymbol{m} \tag{14}$$

where

$$\hat{Y} = \begin{pmatrix} -\lambda_{11}+\mu_{11} & -\lambda_{12}+\mu_{12} & 1/m_1 & -\alpha_{12}+\kappa_{12} \\ -\lambda_{21}+\mu_{21} & -\lambda_{22}+\mu_{22} & \alpha_{12}+\kappa_{12} & 1/m_2 \\ -m_1\omega_1^2 & \beta_{12}-\nu_{12} & -\lambda_{11}-\mu_{11} & -\lambda_{21}-\mu_{21} \\ -\beta_{12}-\nu_{12} & -m_2\omega_2^2 & -\lambda_{12}-\mu_{12} & -\lambda_{22}-\mu_{22} \end{pmatrix}. \tag{15}$$

From (14) it follows that

$$\boldsymbol{m}(t) = \hat{M}(t)\boldsymbol{m}(0) = \exp(t\hat{Y})\boldsymbol{m}(0) \tag{16}$$

where $\boldsymbol{m}(0)$ is given by the initial conditions. The matrix $\hat{M}(t)$ has to fulfil the condition

$$\lim_{t\to\infty} \hat{M}(t) = 0. \tag{17}$$

In order that this limit exists, $\hat{Y}$ must have only eigenvalues with negative real parts.

By direct calculation of $\tilde{L}(q_\kappa q_\mu)$, $\tilde{L}(p_\kappa p_\mu)$ and $\tilde{L}(q_\kappa p_\mu + p_\mu q_\kappa)$, $\kappa, \mu = 1, 2$, we obtain

$$\mathrm{d}\hat{\sigma}/\mathrm{d}t = \hat{Y}\hat{\sigma} + \hat{\sigma}\hat{Y}^{\mathrm{T}} + 2\hat{D} \tag{18}$$

where $\hat{D}$ is the matrix of the diffusion coefficients

$$\hat{D} = \begin{pmatrix} D_{q_1q_1} & D_{q_1q_2} & D_{q_1p_1} & D_{q_1p_2} \\ D_{q_2q_1} & D_{q_2q_2} & D_{q_2p_1} & D_{q_2p_2} \\ D_{p_1q_1} & D_{p_1q_2} & D_{p_1p_1} & D_{p_1p_2} \\ D_{p_2q_1} & D_{p_2q_2} & D_{p_2p_1} & D_{p_2p_2} \end{pmatrix} \tag{19}$$

and $\hat{Y}^{\mathrm{T}}$ the transposed matrix of $\hat{Y}$. The time-dependent solution of (18) can be written as

$$\hat{\sigma}(t) = \hat{M}(t)(\hat{\sigma}(0) - \hat{\Sigma})\hat{M}^{\mathrm{T}}(t) + \hat{\Sigma} \tag{20}$$

where $\hat{M}(t)$ is defined in (16). The matrix $\hat{\Sigma}$ is time independent and solves the static problem (18) ($\mathrm{d}\hat{\sigma}/\mathrm{d}t = 0$):

$$\hat{Y}\hat{\Sigma} + \hat{\Sigma}\hat{Y}^{\mathrm{T}} + 2\hat{D} = 0. \tag{21}$$

Now we assume that the following limit exists for $t \to \infty$:

$$\hat{\sigma}(\infty) = \lim_{t\to\infty} \hat{\sigma}(t). \tag{22}$$

In that case it follows from (20) with (17):

$$\hat{\sigma}(\infty) = \hat{\Sigma}. \tag{23}$$

Inserting (23) into (20) we obtain the basic equations for our purposes:

$$\hat{\sigma}(t) = \hat{M}(t)(\hat{\sigma}(0) - \hat{\sigma}(\infty))\hat{M}^{\mathrm{T}}(t) + \hat{\sigma}(\infty) \tag{24}$$

where

$$\hat{Y}\hat{\sigma}(\infty) + \hat{\sigma}(\infty)\hat{Y}^{\mathrm{T}} = -2\hat{D}. \tag{25}$$

4. The Wigner function and Weyl operator

Finally we want to discuss the time dependence of the Wigner function. This function is defined as

$$f(x_1, x_2, y_1, y_2, t) = \frac{1}{(2\pi\hbar)^4} \iiiint_{-\infty}^{\infty} \exp\left(-\frac{\mathrm{i}}{\hbar}(x_1\eta_1 + x_2\eta_2 - y_1\xi_1 - y_2\xi_2)\right)$$

$$\times \mathrm{Tr}[\rho\tilde{\Phi}_t(W(\xi_1, \xi_2; \eta_1, \eta_2))]\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \tag{26}$$

where the Weyl operator W is defined by ($\xi_1, \xi_2, \eta_1, \eta_2$ real)

$$W(\xi_1, \xi_2; \eta_1, \eta_2) = \exp[\mathrm{i}\hbar^{-1}(\eta_1 q_1 + \eta_2 q_2 - \xi_1 p_1 - \xi_2 p_2)]. \tag{27}$$

Using the method developed by Lindblad (1976a, b) for the one-dimensional case we find for the time development of the Weyl operator the relation

$$\tilde{\Phi}_t(W(\xi_1, \xi_2; \eta_1, \eta_2)) = W(\xi_1(t), \xi_2(t); \eta_1(t), \eta_2(t)) \exp(g(t)). \tag{28}$$

The real functions $\boldsymbol{\xi}(t) = (\xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t))$ and $g(t)$ satisfy the equations of motion:

$$d\boldsymbol{\xi}(t)/dt = \hat{J}\hat{Y}^T\hat{J}^{-1}\boldsymbol{\xi}(t) \tag{29}$$

$$dg(t)/dt = -\hbar^{-2}\boldsymbol{\xi}(t)\hat{J}\hat{D}\hat{J}^{-1}\boldsymbol{\xi}(t) \tag{30}$$

where

$$\hat{J} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{31}$$

Equations (29) and (30) are obtained by inserting the Weyl operator $W(\xi_1, \xi_2; \eta_1, \eta_2)$ into the equation of motion (4) with $\tilde{L}$ defined in (7), (8) and (9). The initial conditions for the coordinates $\xi_1(t)$, $\xi_2(t)$, $\eta_1(t)$ and $\eta_2(t)$ are determined by $\xi_2(0) = \xi_1$, $\xi_2(0) = \xi_2$, $\eta_1(0) = \eta_1$ and $\eta_2(0) = \eta_2$, respectively, and $g(t)$ by $g(0) = 0$. From (29) and (30) we find that $\boldsymbol{\xi}(t)$ is a linear function in the coordinates ξ_1, ξ_2, η_1 and η_2 and $g(t)$ a quadratic function.

The Weyl operator can be used to calculate the time-dependent expectation values $\boldsymbol{m}(t)$ and $\hat{\sigma}(t)$ (see (16) and (20)), since this operator is connected with the coordinates and momenta via the derivatives

$$\begin{aligned} \left.\frac{\partial W}{\partial \xi_i}\right|_{\boldsymbol{\xi}=0} &= -\frac{i}{\hbar} p_i \\ \left.\frac{\partial W}{\partial \eta_i}\right|_{\boldsymbol{\xi}=0} &= \frac{i}{\hbar} q_i \\ \left.\frac{\partial^2 W}{\partial \xi_i\, \partial \xi_j}\right|_{\boldsymbol{\xi}=0} &= -\frac{1}{\hbar^2} p_i p_j \\ \left.\frac{\partial^2 W}{\partial \xi_i\, \partial \eta_j}\right|_{\boldsymbol{\xi}=0} &= \frac{1}{2\hbar^2}(p_i q_j + q_j p_i) \\ \left.\frac{\partial^2 W}{\partial \eta_i\, \partial \eta_j}\right|_{\boldsymbol{\xi}=0} &= -\frac{1}{\hbar^2} q_i q_j. \end{aligned} \tag{32}$$

For example, one obtains by using (32)

$$\sigma_{p_i p_j}(t) = -\hbar^2 \operatorname{Tr}\left(\rho \left. \frac{\partial^2 \tilde{\phi}_t(W)}{\partial \xi_i(0)\, \partial \xi_j(0)} \right|_{\boldsymbol{\xi}(t=0)=0} \right). \tag{33}$$

Equations of this type can be evaluated with the help of (28)-(30) and lead to the same results for $\boldsymbol{m}(t)$ and $\hat{\sigma}(t)$ as given in § 3. With the Weyl operator (28) we can calculate the time development of the Wigner function. For this purpose we use the Fourier transform of the Wigner function at $t = 0$:

$$\begin{aligned} &\operatorname{Tr}\{\rho \exp[i\hbar^{-1}(\eta_1' q_1 + \eta_2' q_2 - \xi_1' p_1 - \xi_2' p_2)]\} \\ &\quad = \iiiint_{-\infty}^{+\infty} \exp[i\hbar^{-1}(x_1\eta_1' + x_2\eta_2' - y_1\xi_1' - y_2\xi_2')] \\ &\quad\quad \times f(x_1, x_2, y_1, y_2, t = 0)\, dx_1\, dx_2\, dy_1\, dy_2. \end{aligned} \tag{34}$$

When this relation is inserted into (26) after the Weyl operator $\tilde{\phi}_t(W)$ is expressed by (28), one can integrate over the coordinates ξ_1, ξ_2, η_1 and η_2 with the following result for the Wigner function:

$$f(x_1, x_2, y_1, y_2, t)$$

$$= \frac{1}{[\det(4\pi\hat{Z})]^{1/2}} \int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} \exp[-\tfrac{1}{4}(\boldsymbol{x} - \boldsymbol{x}'\hat{M}^{\mathrm{T}})\hat{Z}^{-1}(\boldsymbol{x} - \hat{M}\boldsymbol{x}')]$$

$$\times f(x_1', x_2', y_1', y_2', t=0)\, \mathrm{d}x_1'\, \mathrm{d}x_2'\, \mathrm{d}y_1'\, \mathrm{d}y_2' \tag{35}$$

where $\boldsymbol{x} = (x_1, x_2, y_1, y_2)$ and the matrix $\hat{Z}(t)$ is given by

$$\hat{Z}(t) = \int_0^t \hat{M}(t')\hat{D}\hat{M}^{\mathrm{T}}(t')\, \mathrm{d}t'. \tag{36}$$

This definition can be applied in order to rewrite (24):

$$\hat{\sigma}(t) = \hat{M}(t)\hat{\sigma}(0)\hat{M}^{\mathrm{T}}(t) + 2\hat{Z}(t). \tag{37}$$

In the particular case when we set

$$f(x_1, x_2, y_1, y_2, t=0) = \frac{1}{[\det(2\pi\hat{\sigma}(0))]^{1/2}} \exp[-\tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{m}(0))\hat{\sigma}(0)^{-1}(\boldsymbol{x} - \boldsymbol{m}(0))] \tag{38}$$

we obtain from (35)

$$f(x_1, x_2, y_1, y_2, t) = \frac{1}{[\det(2\pi\hat{\sigma}(t))]^{1/2}} \exp[-\tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{m}(t))\hat{\sigma}(t)^{-1}(\boldsymbol{x} - \boldsymbol{m}(t))] \tag{39}$$

which is the well known result for Wigner functions (Wang and Uhlenbeck 1945, Agarwal 1971, Dodonov and Manko 1985).

5. Example for damped oscillators

In order to illustrate the formalism developed in the preceding sections, we present an example of two oscillators, which are coupled by a potential of the form as used for the proton and neutron degrees of freedom in (3), i.e. $\kappa_{12} = 0$, $\mu_{ij} = 0$, $\nu_{12} \neq 0$. In this case the matrix $\hat{Y}$, governing the time development of the expectation values $\boldsymbol{m}(t)$ and $\hat{\sigma}(t)$, becomes

$$\hat{Y} = \begin{pmatrix} -\lambda_{11} & -\lambda_{12} & 1/m_1 & -\alpha_{12} \\ -\lambda_{21} & -\lambda_{22} & \alpha_{12} & 1/m_2 \\ -m_1\omega_1^2 & \beta_{12} - \nu_{12} & -\lambda_{11} & -\lambda_{21} \\ -\beta_{12} - \nu_{12} & -m_2\omega_2^2 & -\lambda_{12} & -\lambda_{22} \end{pmatrix}. \tag{40}$$

For the calculation of the matrix $\hat{M}(t)$ we must diagonalise the matrix $\hat{Y}$ by solving the corresponding secular equation, i.e. $\det(\hat{Y} - z\hat{I}) = 0$, where z is the eigenvalue and $\hat{I}$ is the unit matrix. According to (40) one obtains an equation of fourth order for the eigenvalues z, which can be simply solved only for special examples. In the particular case with $\alpha_{12} = 0$, $\beta_{12} = 0$, $\lambda_{12} = 0$ and $\lambda_{21} = 0$, the secular equation is obtained as

$$[(z + \lambda_{11})^2 + \omega_1^2][(z + \lambda_{22})^2 + \omega_2^2] = (\nu_{12})^2/m_1 m_2. \tag{41}$$

The roots of this equation have the general structure

$$z_1 = -\gamma_+ + \mathrm{i}\omega_+ \qquad z_2 = -\gamma_+ - \mathrm{i}\omega_+ \qquad z_3 = -\gamma_- + \mathrm{i}\omega_- \qquad z_4 = -\gamma_- - \mathrm{i}\omega_-. \tag{42}$$

This can be easily seen for the case $\nu_{12}=0$:

$$\gamma_+ = \lambda_{11} \qquad \gamma_- = \lambda_{22} \qquad \omega_+ = \omega_1 \qquad \omega_- = \omega_2 \tag{43}$$

or for $\lambda_{11}=\lambda_{22}=\lambda$, $\omega_1=\omega_2=\omega(\omega^2>\nu_{12}/(m_1m_2)^{1/2})$

$$\gamma_+ = \gamma_- = \lambda \qquad \omega_\pm^2 = \omega^2 \pm \nu_{12}/(m_1m_2)^{1/2}. \tag{44}$$

Only positive values of γ_+ and γ_- fulfil (17). Applying the eigenvalues z_i of $\hat{Y}$ we can write the time-dependent matrix $\hat{M}(t)$ as follows:

$$M_{mn}(t) = \sum_i N_{mi} \exp(z_i t) N_{in}^{-1} \tag{45}$$

where the matrix $\hat{N}$ represents the eigenvectors of $\hat{Y}$:

$$\sum_n Y_{mn} N_{ni} = z_i N_{mi}. \tag{46}$$

The eigenvalues (42) lead to the following matrix elements M_{mn}:

$$\begin{aligned} M_{mn}(t) = {} & A_{mn}^+ \exp(-\gamma_+ t)\cos(\omega_+ t + \phi_{mn}^+) \\ & + A_{mn}^- \exp(-\gamma_- t)\cos(\omega_- t + \phi_{mn}^-). \end{aligned} \tag{47}$$

The coefficients $A_{mn}^\pm$ and phase shifts $\phi_{mn}^\pm$ are obtained from the eigenvectors of $\hat{Y}$. With the relations $M_{mn}(t=0)=\delta_{mn}$ and $\mathrm{d}M_{mn}(t)/\mathrm{d}t|_{t=0}=Y_{mn}$ we have two equations which the quantities $A_{mn}^\pm$ and $\phi_{mn}^\pm$ have to fulfil:

$$A_{mn}^+ \cos\phi_{mn}^+ + A_{mn}^- \cos\phi_{mn}^- = \delta_{mn} \tag{48a}$$

$$A_{mn}^+(\gamma_+ \cos\phi_{mn}^+ + \omega_+ \sin\phi_{mn}^+) + A_{mn}^-(\gamma_- \cos\phi_{mn}^- + \omega_- \sin\phi_{mn}^-) = -Y_{mn}. \tag{48b}$$

These two equations can be used to eliminate the coefficients $A_{mn}^\pm$. Using (16), (24) and (47) we conclude that the expectation values of the coordinates and momenta decay with the exponential factors $\exp(-\gamma_+ t)$ and $\exp(-\gamma_- t)$, and the matrix elements σ_{mn} with the combined factors $\exp(-2\gamma_+ t)$, $\exp(-2\gamma_- t)$ and $\exp[-(\gamma_+ + \gamma_-)t]$.

Since the matrix elements M_{mn} are in general lengthy expressions, we present here the matrix $\hat{M}(t)$ only for the special and simple case that the oscillators are uncoupled. With the roots given in (43) we obtain

$$\hat{M}(t) = \begin{pmatrix} \exp(-\lambda_{11}t)\cos\omega_1 t & 0 & (1/m_1\omega_1)\exp(-\lambda_{11}t)\sin\omega_1 t & 0 \\ 0 & \exp(-\lambda_{22}t)\cos\omega_2 t & 0 & (1/m_2\omega_2)\exp(-\lambda_{22}t)\sin\omega_2 t \\ -m_1\omega_1\exp(-\lambda_{11}t)\sin\omega_1 t & 0 & \exp(-\lambda_{11}t)\cos\omega_1 t & 0 \\ 0 & -m_2\omega_2\exp(-\lambda_{22}t)\sin\omega_2 t & 0 & \exp(-\lambda_{22}t)\cos\omega_2 t \end{pmatrix} \tag{49}$$

This matrix can be used to evaluate $\hat{\sigma}(t)$ defined by (24) or (37). For example, we find the following expression for $\sigma_{12}=\sigma_{q_1q_2}$ with $\hat{M}(t)$ of (49):

$$\begin{aligned} \sigma_{q_1q_2}(t) = {} & \exp[-(\lambda_{11}+\lambda_{22})t]\Big((\sigma_{q_1q_2}(0)-\sigma_{q_1q_2}(\infty))\cos\omega_1 t\cos\omega_2 t \\ & + \frac{1}{m_1\omega_1}(\sigma_{q_2p_1}(0)-\sigma_{q_2p_1}(\infty))\sin\omega_1 t\cos\omega_2 t \\ & + \frac{1}{m_2\omega_2}(\sigma_{q_1p_2}(0)-\sigma_{q_1p_2}(\infty))\cos\omega_1 t\sin\omega_2 t \\ & + \frac{1}{m_1m_2\omega_1\omega_2}(\sigma_{p_1p_2}(0)-\sigma_{p_1p_2}(\infty))\sin\omega_1 t\sin\omega_2 t\Big) + \sigma_{q_1q_2}(\infty). \end{aligned} \tag{50}$$

Similar expressions are found for the other matrix elements of $\hat{\sigma}(t)$. The matrix elements of $\hat{\sigma}(\infty)$ depend on $\hat{Y}$ and $\hat{D}$ and must be evaluated with (25) or by the relation

$$\hat{\sigma}(\infty) = 2\int_0^\infty \hat{M}(t')\hat{D}\hat{M}^{\mathrm{T}}(t')\,\mathrm{d}t'. \tag{51}$$

As an example we present the value of $\sigma_{q_1q_2}(\infty)$:

$$\begin{aligned}\sigma_{q_1q_2}(\infty) = {} & 2\{[(\lambda_{11}+\lambda_{22})^2+(\omega_1+\omega_2)^2][(\lambda_{11}+\lambda_{22})^2+(\omega_1-\omega_2)^2]\}^{-1}\\ & \times\{(\lambda_{11}+\lambda_{22})[(\lambda_{11}+\lambda_{22})^2+\omega_1^2+\omega_2^2]D_{q_1q_2}\\ & +[(\lambda_{11}+\lambda_{22})^2+\omega_1^2-\omega_2^2]D_{q_2p_1}/m_1\\ & +[(\lambda_{11}+\lambda_{22})^2+\omega_2^2-\omega_1^2]D_{q_1p_2}/m_2+2(\lambda_{11}+\lambda_{22})D_{p_1p_2}/m_1m_2\}.\end{aligned} \tag{52}$$

Similar expressions are obtained for the other matrix elements of $\hat{\sigma}(\infty)$. The diffusion coefficients $D_{q_1q_2}$, $D_{q_1p_2}$, $D_{q_2p_1}$ and $D_{p_1p_2}$ are in general zero for uncoupled oscillators interacting with a usual environment. This has the consequence that the expectation values $\sigma_{q_1q_2}$, $\sigma_{q_1p_2}$, $\sigma_{q_2p_1}$ and $\sigma_{p_1p_2}$ vanish for $t\to\infty$. It is a very interesting point that the general theory of Lindblad allows couplings via the environment between uncoupled oscillators with $\kappa_{12}=0$, $\mu_{ij}=0$, $\nu_{12}=0$. According to the definitions of the parameters in terms of the vectors $\boldsymbol{a}_\kappa$ and $\boldsymbol{b}_\kappa$, the diffusion coefficients above can be different from zero and simulate an interaction between the 'uncoupled' oscillators. In this case a structure of the environment is reflected in the motion of the oscillators.

6. Conclusions

In this paper we have shown the Hamiltonian of the proton and neutron asymmetry degrees of freedom in deep inelastic collisions as an example for two coupled and damped oscillators. For this application the Lindblad theory provides a treatment of damping which is a possible extension of quantum mechanics to open systems. According to this theory we could calculate the damping of the expectation values of coordinates and momenta and the variances as functions of time. The resulting time dependence of the expectation values yields an exponential damping.

The usual limitation of the Lindblad theory is that the damping time is long compared with the characteristic times of the oscillators. This condition is not too well satisfied in deep inelastic collisions as we already pointed out in the introduction. Therefore, we consider the Lindblad theory in an axiomatic manner and accept its parameters as free quantities, fitted to the experimental data of the charge and mass distributions after deep inelastic collisions. The problem arising then is the interpretation of these parameters with the properties of the intrinsic degrees of freedom. For example, the parameters could be related to those obtained from fitting experimental data with the Fokker–Planck equation. Work in this direction is in progress.

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